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Expectation Hypothesis bias: Risk aversion versus Stochastic adjustment

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Abstract

From a long time the classical expectation hypothesis has been challenged from both empirical and theoretical perspective. Still no one could explain entirely the existent bias between expected future spot rates and forward rates, the so called puzzle in the expectation hypothesis

In this work we will address this issue through arbitrage theory, in particular, focusing our attention in a connection of the classical expectation hypothesis to a certain probability measures where the relation between expected future spot rates and forward rates holds. We will approach this applying certain instantaneous spot rate models, verifying in these models that through changes from "real world" probability measure to other probability measures, we will find adjustments that will able us to explain this bias in the expectation hypothesis.

Resumo

Desde algum tempo que a teoria clássica das expectativas tem sido posta em causa tanto numa prespectiva empírica como teórica. Ainda ninguém conseguiu explicar verdadeiramente o desvio que existe entre o valor esperado das futuras taxas spot e as taxas forward, o chamado puzzle da teoria das expectativas.

Neste trabalho iremos abordar este assunto através da teoria da arbitragem, em particular, focando a nossa atenção na ligação entre a teoria clássica das expectativas a uma certa medida de probabilidade onde a relação entre o valor esperado das futuras taxas spot e taxas forward se verifica. Vamos abordar através de modelos de taxas spot instantâneas, verificando nestes modelos que através de mudanças da medida de probabilidade da "vida real" para outras medidas de probabilidade, encontramos ajustes que nos poderão ajudar a explicar o desvio na teoria das expectativas

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Chapter 1

Introduction to Expectation Hypothesis

In this work we will approach the expectation hypothesis through the arbitrage theory. The primary objects in use are the spot rates, forward rates and zero coupon bonds (also known as pure discount bonds). The forward rate is a rate contracted at time t to start in a future time S until the maturity of the contract, T. The spot rate is a rate contracted today for a time interval $[t, T]$. The forward rates and spot rates coincide when the time of contracting the forward rate is equal to the start of the contract over which the forward rate is effective. A zero coupon bond with maturity T, or T-bond, is a contract which guarantees to the holder 1 euro at maturity T (suppose currency is the Euro). We can obtain the present value of a T-bond by discounting it the spot rate at the same maturity T. These financial objects will be fundamental for the results in this work.

The theory of the classical expectation hypothesis postulates that the forward rates are the unbiased predictor of the future spot rates. Several studies about this theory have been published with several empirical and theoretical explanations about the relation between the term structure of interest rates and future rates.

On one hand studies like Longstaff[19] tested the expected hypothesis at the extreme short end of the term structure, using repurchase (repo) rates with maturities measured in days or weeks. He concluded that expectation hypothesis can not be rejected. Note that repo rates are considered the actual cost of holding riskless securities, as referred by Corte et al.[9].

On the other hand, there are studies that reject empirically the classical expectation hypothesis (e.g.: Fama[14], Fama and Bliss[15], Campbell and Shiller[8], Bekaert, Hodrick, Marshall[4], Boudoukh, Richardson and Whitelaw[7]). These empirical studies concluded that there is a bias due to a risk aversion effect. Some of these studies argue about the risk aversion effect to be either constant or time varying. Examples of studies in favour of time varying Fama and Bliss[15], Shaliastovich and Bansal[21], other in favour of constant risk aversion effect Bansal and Yaron[23], Eraker[13], Piazzesi and Schneider[20].

Some theories have been developed to try explain the risk aversion effect such as liquidity preferences, preferred habitat and market segmentation, being the first two the most well known. The definition of liquidity preferences, first introduced by John Keynes[18], is a potentiality or functional tendency which fixes the quantity of money that agents will hold for a fixed interest rate. In others words the amount of money in the economy depends directly of a liquidity function. This theory points that for investors to hold long term securities a premium should be offered and this risk premium increases with maturity in a decreasing proportion. Preferred habitat theory states that the risk premium is not uniformly increasing, and agents are willing to invest if the risk premium received offsets their price.

In terms of the arbitrage theory, this rejection of the classical expectation hypothesis is the same to say that in "real world" probability measure, or physical probability measure P, the expectation hypothesis does not hold because of the risk aversion effect. In other words, the expected future instantaneous spot rate at time T is not equal to the instantaneous forward rate with maturity T in the "real world" probability measure. On the other hand, there is a probability measure equivalent to P, the so called risk neutral probability measure or martingale measure Q, where the risk aversion effect does not influence the value of interest rates. This way we can relate the previous mentioned empirical studies, which rejected the expectation hypothesis, by saying that in a risk neutral world probability measure the relation between future spot rates and forward rates should hold.

In arbitrage theory this argument in not quite true. For the rejection of the expectation hypothesis to be related only with risk aversion effect we should guarantee that interest rates are deterministic. In reality we know they are not, they are stochastic. In arbitrage theory we know that forward rates are the expected future spot rates in another probability measure called T-forward measure that we can obtain changing from risk neutral probability measure to T-forward measure. This way we can say that the so called expectation hypothesis bias exists not only because of the risk aversion effect but also because of the stochastic effect.

Confirming that the risk aversion effect does not explain the bias entirely we refer to an empirical work done by Backus et al[3]. In this work was analyzed the expectation hypothesis with U.S. Treasury bills, and to work the risk aversion effect they have based their economy consumption on a representative agent, with constant relative risk aversion preferences, with one single commodity whose preferences are additively separable over time and obey the expected utility axioms, being the preferences of the representative agent characterized by a expected utility function. Using constant relative risk aversion utility function (CRRA), they concluded that only relative risk aversion factors greater than 8 support the risk aversion effect as an explanation for the expectation hypothesis bias, where common observed factors are lower than these (Hansen and Singleton[16] refer to factors of 1 and 2 has bounds for the relative risk aversion factors).

In this work we will take into account both risk aversion and stochastic effects and try to show that considering the stochastic effect helps explain the puzzling results of Backus et al[3]. We will exploit this through closed form results for the expected value of instantaneous spot rates in the the probability measures P, Q and T-forward, using instantaneous spot rate dynamic models. Through this changes in the probability measure we will be able to find risk aversion adjustment, $RA(t,T)$, and stochastic adjustment, $SA(t,T)$.

We will use two popular instantaneous spot rate models, Vasicek[22] and CIR (Cox Ingersoll and Ross)[12]. These are well-known special cases of affine term structure models (ATS). With this type of models we can easily obtain closed formulas, such as forward rates. Ultimately, from Vasicek and CIR models, we want to compute the closed form of the expected instantaneous spot rates in P, Q and T-forward measure. Note that Vasicek and CIR dynamics are defined in the risk neutral probability measure Q. To obtain the dynamic of these in P measure we need to change from Q measure to P measure via Girsanov theorem which will only affect the drift of both Vasicek and CIR dynamics. After, is just a matter of solving a stochastic differential equation to obtain the expected value in P and Q measure. To obtain the expected value in T-forward measure we will use a result from the arbitrage theory which relates forward rates with this same expected value. With these expected values we will obtain the stochastic and risk aversion adjustments which we will consider the expected bias of the expectation hypothesis. At the end, the sum of these and the forward rate will represent a unbiased predictor of future spot rates in the "real world" probability measure

With this work we hope to contribute to an explanation of the expectation hypothesis or guide into a new direction in solving this puzzle.

Chapter 2

Arbitrage Theory

2.1 Concepts

In this section we introduce definitions of the zero coupon bonds, forward rates, spot rates and instantaneous forward and spot rates.

For the remaining of the work we will adopt notation $p(t,T)$ to define a T-bond at time t with maturity T, $f(t,T)$ to define instantaneous forward rate contracted at time t with maturity T and instantaneous spot rates at time t as $r(t)$. As previous mentioned, a T-bond at the end of the maturity will value $p(T,T) = 1$, for all T. The bond price $p(t,T)$ is a stochastic object with two variables, t and T. For a fixed value of t the bond price will be a function of T which provides prices for bonds of all possible maturities. The graph of this function is called "the bond price curve at t" or "Term Structure at time t". This graph will be differentiable w.r.t. T. For a fixed maturity T, $p(t,T)$ will be a stochastic process function of t This process gives the prices at different times of the bond with fixed maturity T and the trajectory will typically be very irregular.

Suppose that we stand at time t and we have two bonds with maturities S and T, $S < T$. The simple forward rate $L(t;S,T)$ is defined by

$$L(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)} \tag{2.1}$$

while the continuously compounded forward rate $R(t;S,T)$ is defined by

$$R(t;S,T) = -\frac{\ln p(t,T) - \ln p(t,S)}{T-S} \tag{2.2}$$

In the same context, the simple spot rate $L(S,T)$ for the period $[S,T]$ is defined by

$$L(S,T) = -\frac{p(S,T)-1}{(T-S)p(S,T)} \tag{2.3}$$

while the continuously compounded spot rate $R(S,T)$ is defined by

$$R(S,T) = -\frac{\ln p(S,T)}{T-S} \tag{2.4}$$

The instantaneous forward rate with maturity T, contracted at t is defined by

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} \tag{2.5}$$

The instantaneous spot rate at time t is defined by

$$r(t) = f(t,t) \tag{2.6}$$

2.2 Arbitrage Theory

In this subsection we start by defining the money account process to be

$$B(T) = exp\left\{-\int_0^T r(s)\,ds\right\} \tag{2.7}$$

i.e.,

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}$$

The money account can be seen in two ways, first has describing a bank account with stochastic rate of interest, second be seen that investing in it is equivalent to invest in a self-financing "rolling over" trading strategy, which at time t consists entirely of "maturing" the bonds from t to $t + dt$.

We will consider instantaneous spot rate dynamics of the following form

$$dr(t) = a(t, r(t))dt + v(t, r(t))dW(t) \tag{2.8}$$

where W is a Wiener process.

If we assumed that instantaneous spot rates were deterministic than the price of a T-claim $\mathcal{X}$ (we refer to definition of a T-claim to Björk[6]) in a risk neutral world is given by

$$\Pi(t;\mathcal{X}) = e^{-r\cdot(T-t)} E^{Q}_{t,r}\left[\mathcal{X}\right] \tag{2.9}$$

In real life the deterministic risk free rate assumption does not hold because the instantaneous spot rate r is stochastic.

We introduce the general pricing formula of a T-claim $\mathcal{X}$ in a risk neutral valuation, i.e., under the risk neutral probability measure Q

$$\Pi(t;\mathcal{X}) = E^{Q}_{t,r}\left[exp\left\{-\int_t^T r(s)\,ds\right\}\mathcal{X}\right] \tag{2.10}$$

This expression is the same as if we consider the bank account as the numeraire, with $t = 0$.

$$\Pi(t;\mathcal{X}) = E^{Q}_{0,r}\left[\frac{\mathcal{X}}{B(T)}\right] \tag{2.11}$$

For more details about this we refer to Björk[6].

To obtain the price process we need to compute this expected value, but as it is we can not do much about it. In general it is hard to obtain the probability distribution of $\frac{\mathcal{X}}{B(T)}$ (under Q). We can't assume independence between T-claim $\mathcal{X}$ and discount factor $B(T)$ because in most cases r and $\mathcal{X}$ are not independent under Q. We can do something if, instead of having the Bank account $B(t)$ as numeraire of the price process, we have a T-bond (change from Q measure to T-forward measure).

As mentioned before, for the probability measure Q we have the bank account B_t as the numeraire, whereas for the T-Forward measure we have the T-bond $p(t,T)$ as the specific numeraire. For these two we can obtain the same pricing process of a specific T-claim $\mathcal{X}$

$$\Pi(0;X) = B(0)E^{Q}\left[\frac{\mathcal{X}}{B(T)}\right] \tag{2.12}$$

$$\Pi(0;X) = p(0,T)E^{T}\left[\frac{\mathcal{X}}{p(T,T)}\right] \tag{2.13}$$

Now changing from Q to T-Forward measure we will use the Bayes theorem, computing equation (2.13) and obtaining

$$\Pi(0; X) = p(0,T)E^Q\left[\frac{\mathcal{X}}{p(T,T)} \cdot L^T(T)\right] \tag{2.14}$$

where the process L^T is the Radon-Nikodym derivative define by

$$L^T(t) = \frac{dT}{dQ}, on\, F_t \tag{2.15}$$

which is a Q martingale on $\mathcal{F}_t$.
Afterwards we can compute equation (2.13) since $p(T,T) = 1$

$$\Pi(0, \mathcal{X}) = p(0,T)E^T_{t,r}[\mathcal{X}] \tag{2.16}$$

where E^T represents the expectation under the T-Forward measure Q^T.

We can compute equation (2.12) and (2.14)

$$\begin{aligned} B(0)E^Q\left[\frac{\mathcal{X}}{B(T)}\right] &= p(0,T)E^Q\left[\frac{\mathcal{X}}{p(T,T)} \cdot L^T\right] \\ &= p(0,T)E^T\left[\mathcal{X}\right] \end{aligned} \tag{2.17}$$

for all (sufficiently integrable) T-claims $\mathcal{X}$, we deduce that

$$L^T(t) = \frac{p(t,T)}{B(t)p(0,T)}, on\, F_t \tag{2.18}$$

obtaining at the end the Radon Nykodin derivative L^T to change from probability measure Q to T.

Lemma 1. *Assume that for all $T > 0$ we have $r(T)/B(T)$ in $L^1(Q)$, where B is commonly referred as the bank account and r is the instantaneous spot rate dynamics. Then for every T, the process $f(t,T)$ is a Q^T-Martingale for $0 \leq t \leq T$, and in particular we have*

$$E^T_t[r(T)] = f(t,T) \tag{2.19}$$

where T is the forward measure

Proof. With the change of numeraire we can get to the following result

$$\begin{aligned}\Pi(0;X) &= E^Q_{t,r}\left[r(T)\cdot e^{-\int_t^T r(s)\,ds}\right]\\ &= p(t,T)\cdot E^T_{t,r}[r(T)]\end{aligned}$$

Working the expected value of $r(T)$ in the T-forward measure we get

$$\begin{aligned}E^T_{t,r}[r(T)] &= \frac{1}{p(t,T)}E^Q_{t,r}\left[r(T)\cdot e^{-\int_t^T r(s)\,ds}\right]\\ &= -\frac{1}{p(t,T)}E^Q_{t,r}\left[\frac{\partial}{\partial T}\cdot e^{-\int_t^T r(s)\,ds}\right]\\ &= -\frac{1}{p(t,T)}\frac{\partial}{\partial T}E^Q_{t,r}\left[\cdot e^{-\int_t^T r(s)\,ds}\right]\\ &= -\frac{p_T(t,T)}{p(t,T)}\\ &= -\frac{\partial \ln p(t,T)}{\partial T}\\ &= f(t,T)\end{aligned}$$

where $p_T(t,T)$ is $p(t,T)$ derivative in order of maturity T.

□

For more details on this we refer to Björk [6]

2.3 Problem formulation

Most of the studies refer to a bias in the expectation hypothesis related with the risk aversion effect. In arbitrage theory context this is the same as saying forward rates are good predictors of the future spot rates in a risk neutral valuation. The formulation in terms of expected value is

$$E^Q_t[r(T)] = f(t,T) \tag{2.20}$$

From previous sections we mentioned this only holds if the rate $r(T)$ is deterministic. If r is deterministic than we would have the process $L^T(t)$ as

$$
\begin{aligned}
L^T(t) &= \frac{p(t,T)}{B(t)p(0,T)} \\
&= \frac{E_t^Q\left[e^{-\int_t^T r(s)\,ds}\right]}{e^{\int_0^t r(s)\,ds} \cdot E_t^Q\left[e^{-\int_0^T r(s)\,ds}\right]} \\
&= \frac{e^{-\int_t^T r(s)\,ds}}{e^{\int_0^t r(s)\,ds} \cdot e^{-\int_0^T r(s)\,ds}} \\
&= 1
\end{aligned}
$$

Therefore the process L is 1 for all t and thus we would have that Q measure is equal to T-forward measure.

We know for fact that r is stochastic and because of that the forward rates are not good predictors of future spot rates in Q measure. This only holds in T-forward measure.

To address the problem of the expectation hypothesis bias we compute the expected value of $r(T)$ in P, Q and T-forward probability measures. Ultimately the difference between $E^P[r(T)] - E^Q[r(T)]$ and $E^Q[r(T)] - E^T[r(T)]$ will represent the risk aversion adjustment and stochastic adjustment respectively. The focus is than to obtain these adjustments, as well as forward rates. We expect that the sum of these adjustments will correspond to the bias, i.e.,

$$E^P[r(T)] = f(t,T) + SA(t,T) + RA(t,T)$$

To compute these expected values we will assume instantaneous spot rate dynamics for $r(T)$ as equation (2.8) in each probability measure. These dynamic models are defined in probability measure Q and to transform to P measure we will use Girsanov theorem, specifying next what we will use as Girsanov kernel. For the T-forward measure we will obtain the expected value as per equation (2.19), i.e., obtaining the forward rate which we will obtain next in this work.

For the instantaneous spot rate models we will consider the Vasicek and Cox, Ingersoll, Ross (CIR) models, two of the most popular instantaneous spot rate models.

2.4 Affine Term Structure and Bond pricing

An affine term structure hypothesizes that interest rates, at any point in time, are a time invariant linear function of a small set of common factors. The linearity can be seen as criticism because we do not see that in reality. Off course much other models incorporated non linearity but they generally fail in obtaining closed form solutions for the bond prices. On the contrary the linearity assumption of the affine term structure (ATS) models have proven to be remarkably flexible structure for dynamics of bonds.

Independent of any specific model of bond prices, it is always possible to express the price at time t of a zero coupon bond that matures at time T as

$$p(t,T) = E_t^Q \left[exp\left(-\int_t^T r(s)\, ds \right) \right] \tag{2.21}$$

Pricing bonds boils down to specifying the instantaneous spot rate dynamic model under the risk neutral measure Q, making easy for us to derive bond prices. It can be showed that the term structure of interest rates has an affine formulation, assuming that

$$p(t,T) = F(t, r_t; T) \tag{2.22}$$

where F will have the form

$$F(t, r_t; T) = e^{A(t,T) - B(t,T) r_t} \tag{2.23}$$

functions A and B are deterministic. For more details of the ATS formulation we refer to Bjork[6].

Both Vasicek and CIR models are of the ATS type, so we can obtain closed form formulas for bond prices. In the next subsections we present both models and the main formulas.

Through this formulation we can obtain closed forms for bonds in the Vasicek and CIR models.

2.4.1 Vasicek model

The Vasicek instantaneous spot rate model is defined by

$$dr_t = k(\theta - r_t)dt + \sigma dW_t^Q \tag{2.24}$$

Applying the ATS formulation for this model we compute the function A and B and obtain

$$A(t,T) = \left(\theta - \frac{\sigma^2}{2k^2} \right) [B(t,T) - T + t] - \frac{\sigma^2}{4k} B^2(t,T) \tag{2.25}$$

$$B(t,T) = \frac{1}{k}[1 - e^{-k(T-t)}] \tag{2.26}$$

applying A and B to equation (2.23). For more details we refer to Björk([6])

2.4.2 CIR model

The CIR instantaneous spot rate model is defined by

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t^Q \tag{2.27}$$

As in the Vasicek we will have to apply the ATS formulation to equation (2.23) where, $A(t,T) = \ln A_0(T-t)$, and

$$A_0(x) = \left[\frac{2\eta e^{\frac{x}{2}(k+\eta)}}{(\eta+k)(e^{x\eta}-1)+2\eta}\right]^{\frac{2k\theta}{\sigma^2}} \tag{2.28}$$

$$B(x) = \frac{2\left(e^{x\eta}-1\right)}{(\eta+k)(e^{x\eta}-1)+2\eta} \tag{2.29}$$

$$\eta = \sqrt{k^2+2\sigma^2}$$
$$x = T - t$$

For more details we refer to Cox, Ingersoll and Ross([12])

It is known the connection and interdependence between bond prices and the forward rates. We recall it from equation (2.5). Assuming the bond prices to have an ATS formulation (see equation (2.23)) we can easily obtain the forward rate

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} \tag{2.30}$$

$$= -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T}r_t \tag{2.31}$$

This result will be quite handy to obtain, in both Vasicek and CIR, a closed form solution for the forward rates.

Proposition 1 (Vasicek and CIR forward rate)**.** *For the Vasicek model the forward rate is*

$$f(t,T)^{Vasicek} = \frac{e^{-k(T-t)}(1-e^{-k(T-t)})\sigma^2}{2k^2} + \left(1-e^{-k(T-t)}\right)\left(\theta - \frac{\sigma^2}{2k^2}\right) + e^{-k(T-t)}r_t \tag{2.32}$$

For the CIR model the forward rate is

$$f(t,T)^{CIR} = \frac{f_1(t,T) + f_2(r_t,t,T) + f_3(r_t,t,T)}{f_4(r_t,t,T)} \tag{2.33}$$

where

$$\begin{aligned}
f_1(t,T) &= 2\left((e^{2(T-t)(\sqrt{k^2+2\sigma^2})} - 1)k^3\theta + 2\left(e^{2(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k\theta\sigma^2\right)\\
f_2(r_t,t,T) &= 2k^2\left(2e^{(T-t)(\sqrt{k^2+2\sigma^2})}(r_t-\theta) + \theta\left(1 + e^{2(T-t)(\sqrt{k^2+2\sigma^2})}\right)\right)\sqrt{k^2+2\sigma^2}\\
f_3(r_t,t,T) &= 8\sigma^2\sqrt{k^2+2\sigma^2}e^{(T-t)(\sqrt{k^2+2\sigma^2})}r_t\\
f_4(r_t,t,T) &= \sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} + 1\right)\sqrt{k^2+2\sigma^2}\right)^2
\end{aligned}$$

Proof. For the Vasicek model A and B are respectively equations (2.25) and (2.26). For the CIR model A and B are respectively equations (2.28) and (2.29). To obtain forward rates we need to use A and B in equation (2.31).

For the Vasicek model we have that

$$\begin{aligned}
\frac{\partial B(t,T)}{\partial T} &= -\frac{1}{k}\left(\frac{\partial\left(e^{-k(T-t)}\right)}{\partial T}\right)\\
&= e^{-k(T-t)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A(t,T)}{\partial T} &= \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(\frac{\partial B(t,T)}{\partial T} - 1\right) - \frac{2\sigma^2}{4k}B(t,T)\frac{\partial B(t,T)}{\partial T}\\
&= \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(e^{-k(T-t)} - 1\right) - \frac{\sigma^2}{2k}\frac{1}{k}\left(1 - e^{-k(T-t)}\right)e^{-k(T-t)}\\
&= -\frac{e^{-k(T-t)}\left(1 - e^{-k(T-t)}\right)\sigma^2}{2k^2} - \left(1 - e^{-k(T-t)}\right)\left(\theta - \frac{\sigma^2}{2k^2}\right)
\end{aligned}$$

Using equation (2.31) we have the forward rate for Vasicek model

$$f(t,T)^{Vasicek} = \frac{e^{-k(T-t)}\left(1 - e^{-k(T-t)}\right)\sigma^2}{2k^2} + \left(1 - e^{-k(T-t)}\right)\left(\theta - \frac{\sigma^2}{2k^2}\right) + r_t e^{-k(T-t)}$$

Similar as for Vasicek, in the case of CIR model we have

$$\frac{\partial B(x)}{\partial x} = \frac{2\eta e^{k\eta}\left[(\eta+k)\left(e^{x\eta}-1\right)+2\eta\right] - 2\eta e^{k\eta}(\eta+k)\left(e^{x\eta}-1\right)}{\left[(\eta+k)\left(e^{x\eta}-1\right)+2\eta\right]^2}$$

after some simplifications we obtain

$$\frac{\partial B(x)}{\partial x} = \frac{4\eta^2 e^{k\eta}}{\left[(\eta+k)\left(e^{x\eta}-1\right)+2\eta\right]^2}$$

For the case of $A_0(x)$ we start by defining this function as a combination of other two

$$A_0(x) = \left(\frac{\tau_1(x)}{\tau_2(x)}\right)^{\frac{2k\theta}{\sigma^2}}$$

where

$$\begin{aligned}\tau_1(x) &= 2\eta e^{\frac{x}{2}(k+\eta)}\\ \tau_2(x) &= (\eta+k)(e^{x\eta}-1)+2\eta\end{aligned}$$

Computing the derivative of $A(x)$ we have

$$\begin{aligned}\frac{\partial A(x)}{\partial x} &= \frac{\frac{\partial A_0(x)}{\partial x}}{A_0(x)}\\ &= \frac{\frac{2k\theta}{\sigma^2}\left(\frac{\tau_1(x)}{\tau_2(x)}\right)^{\frac{2k\theta}{\sigma^2}-1}\frac{\frac{\partial \tau_1(x)}{\partial x}\tau_2(x)-\tau_1(x)\frac{\partial \tau_2(x)}{\partial x}}{\tau_2^2(x)}}{\left(\frac{\tau_1(x)}{\tau_2(x)}\right)^{\frac{2k\theta}{\sigma^2}}}\end{aligned}$$

after some computations and simplifications we obtain

$$\frac{\partial A(x)}{\partial x} = \frac{2k\theta\left(e^{x\eta}-1\right)}{\left(e^{x\eta}-1\right)k+\eta\left(e^{x\eta}+1\right)}$$

Using equation (2.31), and after some computations, we have the forward rate for CIR model

$$f(t,T)^{CIR} = \frac{f_1(t,T)+f_2(r_t,t,T)+f_3(r_t,t,T)}{f_4(r_t,t,T)}$$

where

$$f_1(t,T) = 2\left((e^{2(T-t)(\sqrt{k^2+2\sigma^2})} - 1)k^3\theta + 2\left(e^{2(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k\theta\sigma^2\right)$$
$$f_2(r_t,t,T) = 2k^2\left(2e^{(T-t)(\sqrt{k^2+2\sigma^2})}(r_t-\theta) + \theta\left(1+e^{2(T-t)(\sqrt{k^2+2\sigma^2})}\right)\right)\sqrt{k^2+2\sigma^2}$$
$$f_3(r_t,t,T) = 8\sigma^2\sqrt{k^2+2\sigma^2}e^{(T-t)(\sqrt{k^2+2\sigma^2})}r_t$$
$$f_4(r_t,t,T) = \sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} + 1\right)\sqrt{k^2+2\sigma^2}\right)^2$$

□

2.5 Physical probability measure and utility functions

We recall the work done by Cox, Ingersoll and Ross[11], and Bick[5], where a formulation of the general term structure was dervied in a risk neutral valuation. On those the specification of the general term structure in a risk neutral valuation was similar as ours. In our formulation, we will use only one state variable, the short rate r_t.

The formulation of the term structure equation in a risk neutral world for the zero coupon bonds is

$$\frac{\partial F}{\partial t}(t,r) + \{\mu^*(t,r_t) - \lambda\}\frac{\partial F}{\partial r}(t,r) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r) - rF(t,r) = 0 \tag{2.34}$$

where $F = p(t,T)$, $\mu^*(t,r_t) - \lambda = \mu(t,R_t)$, where $\mu^*(t,r_t)$ is the drift in the physical measure and λ can be seen as the risk aversion parameter.

Cox et al.[11] refer that this is equivalent of employing an equivalent "risk neutral" economy. They use power utility function, with the constant relative risk aversion parameter to determine λ.

The utility expresses an individual or subjective valuation of money. In other words, it combines both the risk attitude of agents as well as their valuation of money defined in a parametric utility function. As an example, when valuing an asset, depending on the agent's being risk lover or risk averse, will contribute to the price that he is willing to pay it. The utility function plays an important roll in the agent representative theory. Representative agents act in such way that all agents cumulative preferences and actions are the actions of a single one maximizing its expected utility. One of the main assumptions in the representative agent is that market is complete. We will also assume that individuals have homogeneous beliefs and time-additive, state independent utility functions that are strictly concave, increasing and differentiable.

In particular we will use the same utility function as per Cox et al.[11]

$$U(x) = \begin{cases} \frac{x^\epsilon}{\epsilon} & \text{if } \epsilon \neq 0 \\ \ln x & \text{if } \epsilon = 0 \end{cases}$$

In the approach of CIR the market risk parameter λ was linked directly with this utility function, in particular, $\lambda = \epsilon v^2(t,T)$, where v corresponds to the variance of the market portfolio. In the case of Vasicek $v(t,T) = \sigma$ and in the case of CIR $v(t,T) = \sigma\sqrt{r_t}$. We will assume portfolios only based in zero coupon bonds which makes the variance of the zero coupon bonds the portfolio variance.

In the same way, Bick[5] applied a constant relative risk aversion (CRRA) preferences to approach to measure market risk aversion $\lambda = \epsilon v^2(t,T)$, referring that this preferences sustain the Black-Scholes model in equilibrium. In this direction we refer to works of Ait-Sahalia and Lo[2], He and Leland[17].

Summarizing, in a “risk neutral” valuation, our zero coupon bonds term structure equation will be

$$\frac{\partial F}{\partial t}(t,r) + \left\{\mu^*(t,r_t) - \epsilon v^2(t,T)\right\}\frac{\partial F}{\partial r}(t,r) + \frac{1}{2}v^2(t,r_t)\frac{\partial^2 F}{\partial r^2}(t,r) - rF(t,r) = 0 \tag{2.35}$$

where ϵ is the constant relative risk aversion factor of the utility function U stated above. Note that where $\epsilon = 0$, the logarithmic utility, agents are consider to be risk neutral., whereas, values of $\epsilon > 0$ are considered to be risk averse and consequently, for $\epsilon < 0$ investors are assumed to be risk lovers.

Assuming that our bonds follow an ATS model formulation and that $\mu^*(t,r_t) = a(t,r_t) + \epsilon v^2(t,r_t)$, for the short rate formulation in the ”real measure” will be

$$dr_t = \mu^*(t,r_t)dt + v(t,r_t)dW_t^P \tag{2.36}$$

where

$$\begin{aligned} \mu^*(t,r) &= \alpha(t)r + \beta(t) + \epsilon v^2(t,r_t) \\ v^2(t,r) &= \gamma(t)r + \delta(t) \end{aligned} \tag{2.37}$$

If we recall the Girsanov theorem and take a look at the change of the instantaneous spot rate model from Q measure to P measure, we can identify the Girsanov kernel as the constant relative risk aversion factor of the utility function U, $\varphi = \epsilon v(t,T)$.

In particular, for the Vasicek and CIR models the Girsanov kernel will be

$$\varphi^{Vasicek} = \epsilon\sigma \tag{2.38}$$

$$\varphi^{CIR} = \epsilon\sigma\sqrt{r_t} \tag{2.39}$$

Chapter 3

Methodology Application

In this section we will start by computing the expected values of the future spot rates in the physical probability measure P, the risk neutral measure Q and T-forward measure. This is so that we can obtain solutions for the stochastic adjustment, $SA(t,T)$, and risk aversion adjustment $RA(t,T)$, under the hypothesis that the bias is the sum of these two adjustments.

In the case of the expected values of future spot rates in P and Q measures we will compute the stochastic differential equations for both Vasicek and CIR models. We already have the Vasicek and CIR instantaneous spot rate models in risk neutral measure Q. To obtain expected values in the physical measure P we will use the Girsanov theorem, with the Girsanov kernel quoted in the previous section, for both instantaneous spot rate models.

On the other hand, there is no need to compute the expected value of future spot rates in T-forward measure (see equation (2.19)) and we already have computed the forward prices for Vasicek and CIR models (recall equations (2.32) and (2.34) from Proposition 1).

These results for the expected values will be important when to compute the stochastic and risk aversion adjustments.

We start by defining the stochastic adjustment $SA(t,T)$ as an expected value obtained when we change from the T-forward measure to the Q measure, i.e.,

$$E_t^Q[r(T)] = E_t^T[r(T)] + SA(t,T) \tag{3.1}$$

$$\Leftrightarrow$$

$$SA(t,T) = E_t^Q[r(T)] - E_t^T[r(T)] \tag{3.2}$$

Similarly we define the risk aversion adjustment $RA(t,T)$ as an expected value obtained when we change from the Q measure to the P measure.

$$E_t^P[r(T)] = E_t^Q[r(T)] + RA(t,T) \tag{3.3}$$
$$\Leftrightarrow$$
$$RA(t,T) = E_t^P[r(T)] - E_t^Q[r(T)] \tag{3.4}$$

3.1 Risk adjustment

From section 2.4 we know the instantaneous spot rate dynamics in risk neutral measure Q, for both Vasicek and CIR models (equations (2.24) and (2.27) respectively). To obtain the closed form solution for the expected value we need to solve the differential equation in the drift of this equation.

Proposition 2 (Vasicek and CIR risk adjustment)**.** *For the Vasicek model the risk adjustment is*

$$RA^{Vasicek}(t,T) = \frac{\epsilon\sigma^2}{k}\left(1 - e^{-k(T-t)}\right) \tag{3.5}$$

For the CIR model the risk adjustment is

$$RA^{CIR}(t,T) = \frac{\epsilon\theta\sigma^2 - e^{-k(T-t)}(r_t - \theta)(k - \epsilon\sigma^2) + e^{(\epsilon\sigma^2 - k)(T-t)}(k(r_t - \theta) - r_t\epsilon\sigma^2)}{k - \epsilon\sigma^2} \tag{3.6}$$

Proof. For the Vasicek model computations of the expected value of the instantaneous spot rate in Q measure we assume an integrating factor $u(s) = e^{ks}$. Using the general equation for the integrating factor (refer to equation (1.35) from Costa[10]), we have

$$\begin{aligned} E_t^Q[r(T)] &= e^{-kT}e^{kt}r(t) + e^{-kT}\int_t^T e^{ks}k\theta ds \\ &= r(t)e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right) \end{aligned} \tag{3.7}$$

From section 2.5 we also know the instantaneous spot rate dynamics but in risk neutral measure P. We also assume an integrating factor $u(s) = e^{ks}$ and again use the general equation for the integrating factor (refer to equation (1.35) from Costa[10])

$$\begin{aligned} E_t^P[r(T)] &= e^{-kT}e^{kt}r(t) + e^{-kT}\int_t^T e^{ks}\left(k\theta + \epsilon\sigma^2\right) ds \\ &= r(t)e^{-k(T-t)} + \frac{k\theta + \epsilon\sigma^2}{k}\left(1 - e^{-k(T-t)}\right) \end{aligned} \tag{3.8}$$

Similar as in the case of Vasicek model, for the CIR model we assume an integrating factor $u(s) = e^{ks}$ for the computations of the expected value of the instantaneous spot rate in Q measure. Using the general equation for the integrating factor (refer to equation (1.35) from Costa[10]), we have

$$\begin{aligned} E_t^Q[r(T)] &= e^{-kT}e^{kt}r(t) + e^{-kT}\int_t^T e^{ks}k\theta ds \\ &= r(t)e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right) \end{aligned} \tag{3.9}$$

From section 2.5 we also know the instantaneous spot rate model in “real world” measure P. We also assume an integrating factor, but in here needs to be slightly different, $u(s) = e^{(k-\epsilon\sigma^2)s}$. Again use the general equation for the integrating factor (refer to equation (1.35) from Costa[10])

$$\begin{aligned} E_t^P[r(T)] &= e^{-(k-\epsilon\sigma^2)T}e^{(k-\epsilon\sigma^2)t}r(t) + e^{-(k-\epsilon\sigma^2)T}\int_t^T k\theta e^{(k-\epsilon\sigma^2)s}ds \\ &= r(t)e^{-(k-\epsilon\sigma^2)(T-t)} + \frac{k\theta}{(k-\epsilon\sigma^2)}\left(1 - e^{-(k-\epsilon\sigma^2)(T-t)}\right) \end{aligned} \tag{3.10}$$

With the results above we can compute the risk adjustment for the Vasicek model

$$\begin{aligned} RA^{Vasicek}(t,T) &= E_{t,\epsilon}^P[r[T]] - E_t^Q[r[T]] \\ &= \frac{\epsilon\sigma^2}{k}\left(1 - e^{-k(T-t)}\right) \end{aligned}$$

and the risk adjustment for the CIR model

$$\begin{aligned} RA(t,T)^{CIR} &= E_{t,\epsilon}^P[r[T]] - E_t^Q[r[T]] \\ &= \frac{\epsilon\theta\sigma^2 - e^{-k(T-t)}(r_t-\theta)(k-\epsilon\sigma^2) + e^{(\epsilon\sigma^2-k)(T-t)}(k(r_t-\theta) - r_t\epsilon\sigma^2)}{k-\epsilon\sigma^2} \end{aligned}$$

$\square$

3.2 Stochastic adjustment

Proposition 3 (Vasicek and CIR stochastic adjustment)**.** *For the Vasicek model the risk adjustment is*

$$SA^{Vasicek}(t,T) = \frac{e^{-2kT}(e^{kt} - e^{kT})^2\sigma^2}{2k^2} \tag{3.11}$$

For the CIR model the risk adjustment is

$$SA^{CIR}(t,T) = SA_1(t,T) + SA_2(t,T) + SA_3(t,T) + SA_4(t,T) \tag{3.12}$$

where

$$SA_1(t,T) = r_t e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right)$$

$$SA_2(t,T) = -\frac{2\left((e^{2(T-t)(\sqrt{k^2+2\sigma^2})} - 1)k^3\theta + 2\left(e^{2(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k\theta\sigma^2\right)}{\sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} + 1\right)\sqrt{k^2+2\sigma^2}\right)^2}$$

$$SA_3(t,T) = -\frac{2k^2\left(2e^{(T-t)(\sqrt{k^2+2\sigma^2})}(r_t - \theta) + \theta\left(1 + e^{2(T-t)(\sqrt{k^2+2\sigma^2})}\right)\right)\sqrt{k^2+2\sigma^2}}{\sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} + 1\right)\sqrt{k^2+2\sigma^2}\right)^2}$$

$$SA_4(t,T) = -\frac{+8r_t\sigma^2\sqrt{k^2+2\sigma^2}e^{(T-t)(\sqrt{k^2+2\sigma^2})}}{\sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} - 1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})} + 1\right)\sqrt{k^2+2\sigma^2}\right)^2}$$

Proof. Here we also use the expected value of instantaneous spot rate in risk neutral measure Q obtain in section 3.1 for both Vasicek and CIR models (equations (3.7) and (CIR expected Q) respectively.

For the Vasicek model the stochastic adjustment is

$$\begin{aligned}
SA(t,T)^{Vasicek} &= E^Q_{t,\epsilon}[r[T]] - E^T_t[r[T]] \\
&= E^Q_{t,\epsilon}[r[T]] - f(t,T)^{Vasicek} \\
&= \frac{e^{-2kT}(e^{kt} - e^{kT})^2\sigma^2}{2k^2}
\end{aligned}$$

For the CIR model the stochastic adjustment is

$$\begin{aligned}
SA(t,T)^{CIR} &= E^Q_{t,\epsilon}[r[T]] - E^T_t[r[T]] \\
&= E^Q_{t,\epsilon}[r[T]] - f(t,T)^{CIR} \\
&= E^Q_{t,\epsilon}[r[T]] - \left(\frac{f_1(t,T) + f_2(r_t,t,T) + f_3(r_t,t,T)}{f_4(r_t,t,T)}\right) \\
&= SA_1(t,T) + SA_2(t,T) + SA_3(t,T) + SA_4(t,T)
\end{aligned}$$

where

$$E^Q_{t,\epsilon}[r[T]] \equiv SA_1(t,T) = r_t e^{-k(T-t)} + \theta\left(1 - e^{-k(T-t)}\right)$$

and

$$\begin{aligned} SA_2(t,T) &= -\frac{f_1(t,T)}{f_4(t,T)} \\ &= -\frac{2\left((e^{2(T-t)(\sqrt{k^2+2\sigma^2})}-1)k^3\theta + 2\left(e^{2(T-t)(\sqrt{k^2+2\sigma^2})}-1\right)k\theta\sigma^2\right)}{\sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})}-1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})}+1\right)\sqrt{k^2+2\sigma^2}\right)^2} \end{aligned}$$

$$\begin{aligned} SA_3(t,T) &= -\frac{f_2(t,T)}{f_4(t,T)} \\ &= -\frac{2k^2\left(2e^{(T-t)(\sqrt{k^2+2\sigma^2})}(r_t-\theta) + \theta\left(1+e^{2(T-t)(\sqrt{k^2+2\sigma^2})}\right)\right)\sqrt{k^2+2\sigma^2}}{\sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})}-1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})}+1\right)\sqrt{k^2+2\sigma^2}\right)^2} \end{aligned}$$

$$\begin{aligned} SA_4(t,T) &= -\frac{f_3(t,T)}{f_4(t,T)} \\ &= -\frac{+8r_t\sigma^2\sqrt{k^2+2\sigma^2}e^{(T-t)(\sqrt{k^2+2\sigma^2})}}{\sqrt{k^2+2\sigma^2}\left(\left(e^{(T-t)(\sqrt{k^2+2\sigma^2})}-1\right)k + \left(e^{(T-t)(\sqrt{k^2+2\sigma^2})}+1\right)\sqrt{k^2+2\sigma^2}\right)^2} \end{aligned}$$

□

3.3 Summary

With these results we have enough to compute amounts for the expected bias in the expectation hypothesis, the stochastic adjustment and risk adjustment.

Our hypothesis is to say that the expected value of the future short rates is given by

$$E^P_t[r(T)] = f(t,T) + bias(t,T) \tag{3.13}$$

where

$$bias(t,T) = RA(t,T) + SA(t,T) \tag{3.14}$$

Chapter 4

Results

4.1 Vasicek model results

Next we present some results for the Vasicek model. For these the parameters k, σ and θ used were obtained from Zeytun and Gupta[24]. The purpose is to give a sense of the values of the stochastic adjustment (equation (3.11)) and risk aversion adjustment (RA equation (3.5)), and how much they are responsible for what we define as expectation hypothesis bias. Along side we include the forward rate value obtained from equation (2.32), the expected instantaneous spot rate in P measure (3.8), the percentage of the expectation hypothesis bias in this expected spot rate in P measure ($\frac{bias}{E^P[r(T)]}$), and the percentage of each adjustments in the so called bias ($\frac{SA(t,T)}{bias}$ and $\frac{RA(t,T)}{bias}$). All of the values presented in the subtables (b) to (f) are in percentage (%).

Table 4.1 refers to parameters $k = 0.25$, $\sigma = 0.01$ and $\theta = 0.1$, varying the constant relative risk aversion factor ϵ. The subtable (a) contains the parameters used.

	Value
r_t	2,5%
θ	0,1
σ	0,01
k	0,25

(a) Parameters Table

	1	2	5	10
f(t,T)	4,16	5,44	7,81	9,32
$E^P[R(T)]$	4,16	5,45	7,85	9,38
SA(t,T)	0	0,01	0,04	0,07
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	0,09	0,23	0,52	0,72
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	4,16	5,44	7,81	9,32
$E^P[R(T)]$	4,16	5,46	7,87	9,4
SA(t,T)	0	0,01	0,04	0,07
RA(t,T)	0	0,01	0,01	0,02
bias weight in E[r(T)]	0,2	0,37	0,7	0,91
SA weight in bias	47,05	61,1	74,03	78,62
RA weight in bias	52,88	38,97	26	21,46

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	4,16	5,44	7,81	9,32
$E^P[R(T)]$	4,17	5,47	7,88	9,42
SA(t,T)	0	0,01	0,04	0,07
RA(t,T)	0,01	0,02	0,03	0,04
bias weight in E[r(T)]	0,31	0,52	0,88	1,11
SA weight in bias	30,77	43,97	58,76	64,73
RA weight in bias	69,18	55,73	41,12	35,24

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	4,16	5,44	7,81	9,32
$E^P[R(T)]$	4,18	5,48	7,91	9,46
SA(t,T)	0	0,01	0,04	0,07
RA(t,T)	0,02	0,03	0,06	0,07
bias weight in E[r(T)]	0,52	0,8	1,24	1,49
SA weight in bias	18,11	28,23	41,64	47,86
RA weight in bias	81,87	71,8	58,38	52,12

(e) $\epsilon = 2$

	1	2	5	10
f(t,T)	4,16	5,44	7,81	9,32
$E^P[R(T)]$	4,2	5,53	7,99	9,57
SA(t,T)	0	0,01	0,04	0,07
RA(t,T)	0,04	0,08	0,14	0,18
bias weight in E[r(T)]	1,14	1,65	2,29	2,62
SA weight in bias	8,13	13,6	22,2	26,85
RA weight in bias	91,85	86,42	77,8	73,14

(f) $\epsilon = 5$

Table 4.1: Vasicek ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.01$, k = 0.25

Table 4.2 refers to parameters $k = 0.25$, $\sigma = 0.05$ and $\theta = 0.1$, varying the constant relative risk aversion factor ϵ.

	Value
r_t	2,5%
θ	0,1
σ	0,05
k	0,25

(a) Parameters Table

	1	2	5	10
f(t,T)	4,06	5,14	6,83	7,7
$E^P[R(T)]$	4,16	5,45	7,85	9,38
SA(t,T)	0,1	0,31	1,02	1,69
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	2,35	5,68	12,97	17,96
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	4,06	5,14	6,83	7,7
$E^P[R(T)]$	4,27	5,65	8,21	9,84
SA(t,T)	0,1	0,31	1,02	1,69
RA(t,T)	0,11	0,2	0,36	0,46
bias weight in E[r(T)]	4,88	8,97	16,75	21,78
SA weight in bias	46,94	61,14	74,05	78,6
RA weight in bias	53,06	38,84	25,94	21,41

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	4,06	5,14	6,83	7,7
$E^P[R(T)]$	4,38	5,84	8,56	10,3
SA(t,T)	0,1	0,31	1,02	1,69
RA(t,T)	0,22	0,39	0,71	0,92
bias weight in E[r(T)]	7,28	12,03	20,22	25,27
SA weight in bias	30,67	44,04	58,8	64,74
RA weight in bias	69,33	55,96	41,2	35,26

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	4,06	5,14	6,83	7,7
$E^P[R(T)]$	4,6	6,24	9,28	11,22
SA(t,T)	0,1	0,31	1,02	1,69
RA(t,T)	0,44	0,79	1,43	1,84
bias weight in E[r(T)]	11,74	17,58	26,35	31,38
SA weight in bias	18,11	28,24	41,64	47,86
RA weight in bias	81,89	71,76	58,36	52,14

(e) $\epsilon = 2$

	1	2	5	10
f(t,T)	4,06	5,14	6,83	7,7
$E^P[R(T)]$	5,27	7,42	11,42	13,97
SA(t,T)	0,1	0,31	1,02	1,69
RA(t,T)	1,11	1,97	3,57	4,59
bias weight in E[r(T)]	22,87	30,69	40,16	44,9
SA weight in bias	8,13	13,6	22,2	26,86
RA weight in bias	91,87	86,4	77,8	73,14

(f) $\epsilon = 5$

Table 4.2: Vasicek ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k = 0.25

We refer to Appendix subsection 6.1 for other results of the Vasicek model where we change the parameter θ.

As expected, the bias increases with the increase of the maturity.

Comparing tables we can check that when σ increase forward rates decrease but expected instantaneous spot rate under P measure increases (for instance, compare subtable 4.1d with table 4.2d). As a result the bias increases, being the stochastic effect the one with the biggest impact, although we see an increase of the risk aversion effect as we would have expected.

By looking at subtables 4.1d and 4.2d we can see that the weight of stochastic and risk aversion adjustment in the bias does not change.

We can also see that when we increase k, both forward rates and expected instantaneous spot rate increases, although the expectations hypothesis bias decreases (for instance, compare table 4.1c with table 6.1c, this last one in the Appendix subsection 6.1).

4.2 CIR model results

Next we present some results for the CIR model. As per Vasicek, parameters k, σ and θ used were obtained from Zeytun and Gupta[24] (in the case of CIR model, the volatility parameter σ is in a different proportion when compared with the one in Vasicek model). The purpose is to give a sense of the values of the stochastic adjustment (equation (3.12)) and risk aversion adjustment (equation (3.6)), and how much they are responsible for what we define as expectation hypothesis bias. Along side we include the forward rate value obtained from equation (2.34), the expected instantaneous spot rate in P measure (3.10), the percentage of the expectation hypothesis bias in this expected spot rate in P measure ($\frac{bias}{E^P[r(T)]}$), and the percentage of each adjustments in the so called bias ($\frac{SA(t,T)}{bias}$ and $\frac{RA(t,T)}{bias}$). All of the values presented in the tables (b) to (f) are in percentage (%).

Table 4.3 refers to parameters $k = 0.25$, $\sigma = 0.05$ and $\theta = 0.1$, varying the constant relative risk aversion factor ϵ. The subtable (a) contains the parameters used.

	Value
r_t	2,5%
θ	0,1
σ	0,05
k	0,25

(a) Parameters Table

	1	2	5	10
f(t,T)	4,16	5,44	7,8	9,26
$E^P[R(T)]$	4,16	5,45	7,85	9,38
SA(t,T)	0	0,01	0,06	0,13
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	0,07	0,21	0,7	1,35
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	4,16	5,44	7,8	9,26
$E^P[R(T)]$	4,16	5,46	7,87	9,42
SA(t,T)	0	0,01	0,06	0,13
RA(t,T)	0	0,01	0,02	0,04
bias weight in E[R(T)]	0,16	0,36	0,98	1,75
SA weight in bias	44,53	58,26	71,24	76,79
RA weight in bias	55,47	41,74	28,76	23,21

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	4,16	5,44	7,8	9,26
$E^P[R(T)]$	4,17	5,47	7,9	9,46
SA(t,T)	0	0,01	0,06	0,13
RA(t,T)	0,01	0,02	0,04	0,08
bias weight in E[r(T)]	0,25	0,52	1,26	2,15
SA weight in bias	28,93	40,92	55,26	62,25
RA weight in bias	71,07	59,08	44,74	37,75

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	4,16	5,44	7,8	9,26
$E^P[R(T)]$	4,17	5,48	7,94	9,54
SA(t,T)	0	0,01	0,06	0,13
RA(t,T)	0,02	0,03	0,09	0,15
bias weight in E[R(T)]	0,44	0,82	1,83	2,95
SA weight in bias	16,82	25,71	38,07	45,02
RA weight in bias	83,18	74,29	61,93	54,98

(e) $\epsilon = 2$

	1	2	5	10
f(t,T)	4,16	5,44	7,8	9,26
$E^P[R(T)]$	4,2	5,53	8,08	9,78
SA(t,T)	0	0,01	0,06	0,13
RA(t,T)	0,04	0,08	0,23	0,4
bias weight in E[R(T)]	0,97	1,72	3,5	5,34
SA weight in bias	7,48	12,11	19,54	24,29
RA weight in bias	92,52	87,89	80,46	75,71

(f) $\epsilon = 5$

Table 4.3: CIR ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k = 0.25

Table 4.4 refers to parameters $k = 0.25$, $\sigma = 0.25$ and $\theta = 0.1$. The difference between the tables is the constant relative risk aversion factor ϵ which varies between 0 and 5.

	Value
r_t	2,5%
θ	0,1
σ	0,25
k	0,25

(a) Parameters Table

	1	2	5	10
f(t,T)	4,08	5,18	6,72	7,25
$E^P[R(T)]$	4,16	5,45	7,85	9,38
SA(t,T)	0,08	0,27	1,13	2,13
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	1,81	5,02	14,39	22,73
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	4,08	5,18	6,72	7,25
$E^P[R(T)]$	4,25	5,66	8,44	10,43
SA(t,T)	0,08	0,27	1,13	2,13
RA(t,T)	0,1	0,21	0,59	1,04
bias weight in E[r(T)]	4,01	8,59	20,34	30,46
SA weight in bias	44,08	56,26	65,82	67,17
RA weight in bias	55,91	43,72	34,18	32,83

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	4,08	5,18	6,72	7,25
$E^P[R(T)]$	4,35	5,89	9,09	11,67
SA(t,T)	0,08	0,27	1,13	2,13
RA(t,T)	0,19	0,44	1,24	2,29
bias weight in E[r(T)]	6,17	12,07	26,07	37,87
SA weight in bias	28	38,54	47,68	48,25
RA weight in bias	72	61,46	52,32	51,75

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	4,08	5,18	6,72	7,25
$E^P[R(T)]$	4,56	6,37	10,63	14,99
SA(t,T)	0,08	0,27	1,13	2,13
RA(t,T)	0,4	0,92	2,78	5,6
bias weight in E[r(T)]	10,37	18,74	36,79	51,61
SA weight in bias	15,9	22,94	28,89	27,58
RA weight in bias	84,09	77,06	71,11	72,42

(e) $\epsilon = 2$

	1	2	5	10
f(t,T)	4,08	5,18	6,72	7,25
$E^P[R(T)]$	5,24	8,16	18,09	39,4
SA(t,T)	0,08	0,27	1,13	2,13
RA(t,T)	1,08	2,71	10,24	30,02
bias weight in E[r(T)]	22,08	36,54	62,85	81,6
SA weight in bias	6,49	9,18	9,94	6,63
RA weight in bias	93,51	90,82	90,06	93,36

(f) $\epsilon = 5$

Table 4.4: CIR ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.25$, k = 0.25

We refer to Appendix subsection 6.2 for other results for the CIR model where we change the parameters k and θ.

As in the case of Vasicek, the expectation hypothesis bias increases with the increase of the maturity.

Like in the Vasicek model, with the increase of σ we can see a decrease in the forward rate and an increase in the expected instantaneous spot rate which increases the expectation hypothesis bias, being the stochastic effect the one with the biggest increase. We can see this by comparing tables 4.3d with table 4.4d.

With the increase of k and θ we see that forward rates and expected instantaneous spot rates increases. Although, the increasing θ influences little the expectation hypothesis bias (we can see this by comparing table 4.3e with table 6.3e, this last one in the Appendix section). On the other hand, for the increase of k, we see a decrease in the expectation hypothesis bias, as per Vasicek model (we can see this by comparing table 4.3e with table 6.2e, this last one in the Appendix subsection 6.2).

4.3 Implicit risk aversion

The main goal of this work is to show that the bias of the expectation hypothesis is not only due to a risk aversion effect, but also with a stochastic effect.

On one hand, some consider the expectation hypothesis bias as to be related only with risk aversion effect

$$bias \equiv RA(t,T)^* \tag{4.1}$$

where we consider $RA(t,T)^*$ the implicit risk aversion. On the other hand we say the expectation hypothesis bias is a combination of risk aversion effect and stochastic effect

$$bias \equiv RA(t,T) + SA(t,T) \tag{4.2}$$

In Backus et al.[3] was considered that the bias is only due to a risk aversion effect. Although they found that this to hold the constant relative risk aversion (CRRA) factor needed to be 8 or greater, opposite to other studies (Hansen and Singleton[16]) that consider a normal CRRA factor between 1 and 2.

Our approach is that the high CRRA factors found by Backus et al. can be explained if we consider a stochastic effect and a risk aversion effect but with suitable CRRA factor values.

By comparing equations (4.1) and (4.2), we want to obtain a CRRA factor ϵ^* (of equation 4.1) as a function of ϵ (of equation 4.2).

$$RA(t,T)^* = RA(t,T) + SA(t,T) \tag{4.3}$$

Ultimately we want to see if for values of $\epsilon \in [1,2]$, we can find a values of ϵ^* greater than 8 as in Backus et al.[3]

4.3.1 Vasicek implicit risk aversion

Lemma 2. *In the Vasicek model the calculation for the implicit constant relative risk aversion ϵ^* is*

$$RA(t,T)^* = RA(t,T) + SA(t,T) \iff \epsilon^* = \frac{e^{-k(T-t)}\left(-1 + e^{k(T-t)} + 2\epsilon k e^{k(T-t)}\right)}{2k} \tag{4.4}$$

Next we present some results for the implicit CRRA factor ϵ^*

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,4424	1,4424	2,4424	3,4424	4,4424	5,4424
T=2	0,7869	1,7869	2,7869	3,7869	4,7869	5,7869
T=5	1,427	2,427	3,427	4,427	5,427	6,427
T=10	1,8358	2,8358	3,8358	4,8358	5,8358	6,8358
T=20	1,9865	2,9865	3,9865	4,9865	5,9865	6,9865

Table 4.5: Vasicek: Values of ϵ^* considering k = 0.25

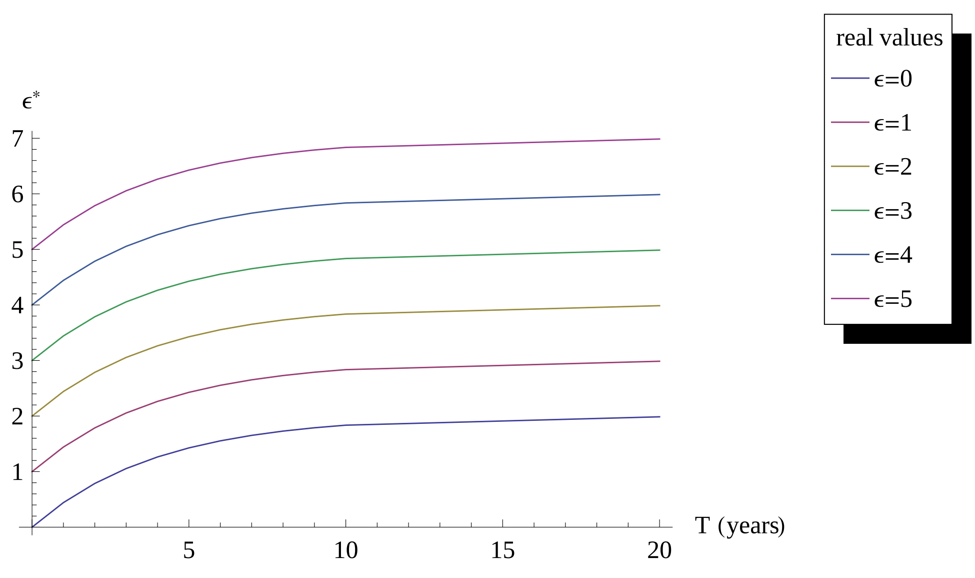

Figure 4.1: Graphic of ϵ^* considering that bias is only risk aversion effect: k=0.25

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,4975	1,4975	2,4975	3,4975	4,4975	5,4975
T=2	0,9901	1,9901	2,9901	3,9901	4,9901	5,9901
T=5	2,4385	3,4385	4,4385	5,4385	6,4385	7,4385
T=10	4,7581	5,7581	6,7581	7,7581	8,7581	9,7581
T=20	9,0635	10,0635	11,0635	12,0635	13,0635	14,0635

Table 4.6: Vasicek: Values of ϵ^* considering k = 0.01

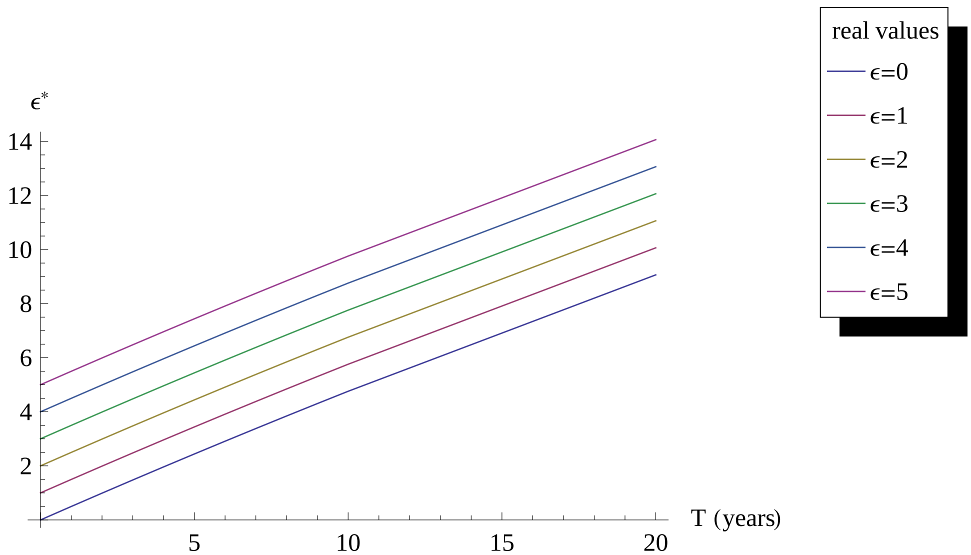

Figure 4.2: Graphic of ϵ^* considering that bias is only risk aversion effect: k=0.01

We refer to Appendix subsection 6.3 for other results for implicit risk aversion of the Vasicek model where we change the parameter k.

With the decrease of k we see that the ϵ^* increases and with that a closer approximation to the CRRA factors referred by Backus et al.[3], for higher maturities.

4.3.2 CIR implicit risk aversion

For the CIR model the calculations for the constant relative risk aversion ϵ^* are obtained numerically by approximating this by Newton–Raphson method (see Alves[1] for more details).

The equation to be determined numerically was

$$RA(t,T)^* = RA(t,T) + SA(t,T) \tag{4.5}$$

where equation $SA(t,T)$ refers to equation (3.12) and equations $RA(t,T)$ and $RA(t,T)^*$ refer to equation (3.6), but with different constant relative risk aversion parameters ϵ and ϵ^* respectively.

Next we present some results for the implicit CRRA factor ϵ^*

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,476	1,4748	2,4737	3,4725	4,4714	5,4703
T=2	0,9075	1,9032	2,899	3,8948	4,8906	5,8864
T=5	1,9775	2,9562	3,935	4,9141	5,8934	6,8729
T=10	3,1938	4,132	5,0713	6,0116	6,9529	7,8952
T=20	4,359	5,2162	6,0771	6,9418	7,81	8,6818

Table 4.7: CIR: Values of ϵ^*: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k = 0.05

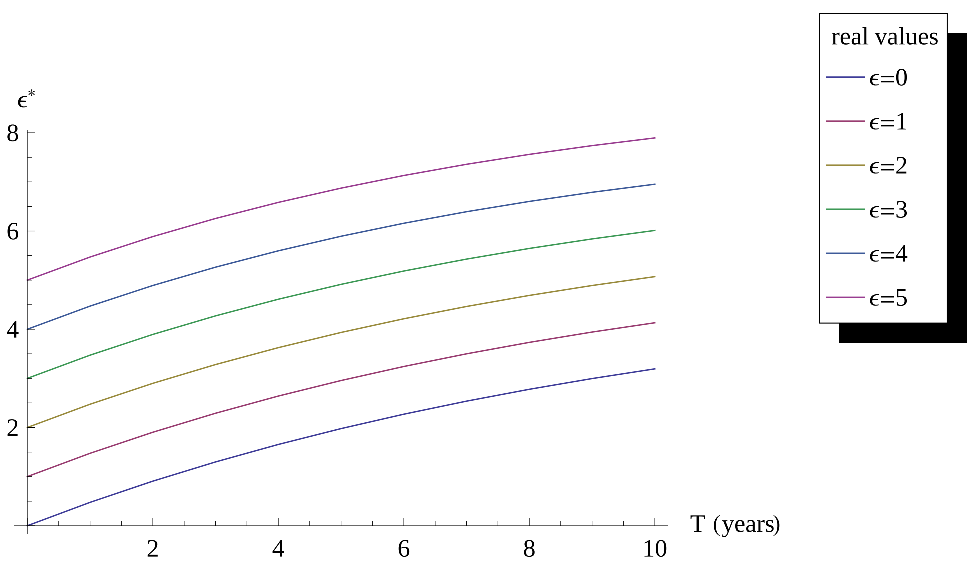

Figure 4.3: Graphic of ϵ^* considering that bias is only risk aversion effect: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k = 0.05

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,3409	1,3243	2,3084	3,2932	4,2787	5,2649
T=2	0,5173	1,474	2,4337	3,3964	4,3618	5,3299
T=5	0,7044	1,5963	2,5009	3,4174	4,3452	5,2834
T=10	0,7373	1,5856	2,4544	3,3438	4,2531	5,1811
T=20	0,7342	1,5693	2,4245	3,3009	4,2	5,1225

Table 4.8: CIR: Values of ϵ^*: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.25$, k = 0.5

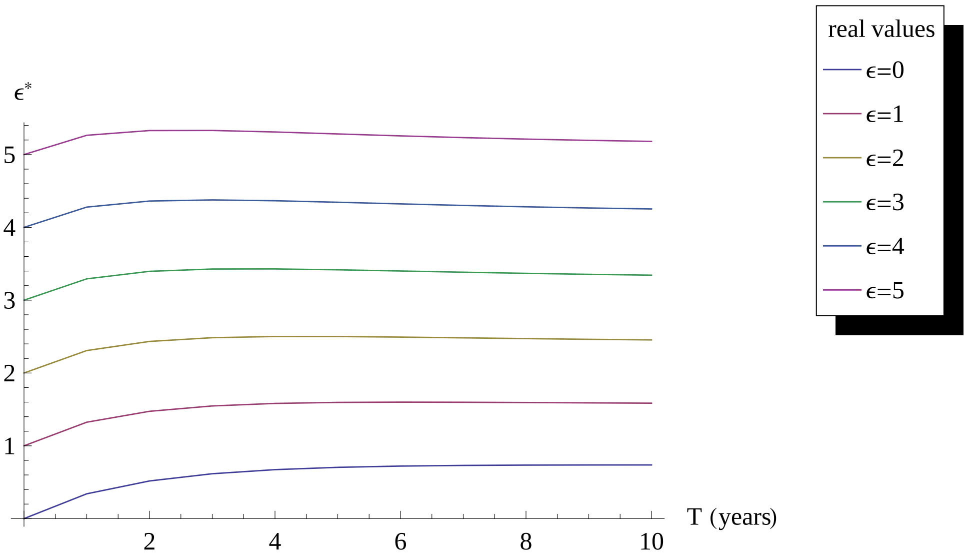

Figure 4.4: Graphic of ϵ^* considering that bias is only risk aversion effect: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.25$, k = 0.5

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,4949	1,4946	2,4943	3,4941	4,4938	5,4934
T=2	0,9792	1,978	2,9767	3,9756	4,9744	5,9731
T=5	2,3629	3,3558	4,3487	5,3417	6,3346	7,3275
T=10	4,4174	5,3918	6,3663	7,3411	8,3159	9,2907
T=20	7,5416	8,4607	9,3807	10,3015	11,223	12,1452

Table 4.9: CIR: Values of ϵ^*: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.025$, k = 0.01

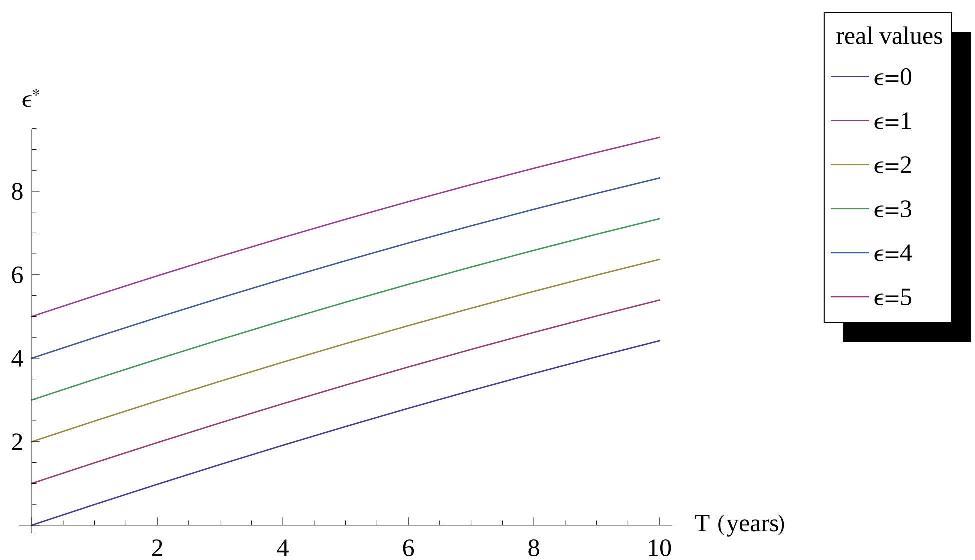

Figure 4.5: Graphic of ϵ^* considering that bias is only risk aversion effect: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.025$, k = 0.01

We refer to Appendix subsection 6.4 for other results for implicit risk aversion of the Vasicek model where we change the parameters k and θ.

With an increase of k we see a decrease of ϵ^*, as in the Vasicek model.

On the contrary of Vasicek model, the σ effects the value of ϵ^*. As for parameter k, with the increase of σ we see a decrease of ϵ^*.

We can only find results closer to the ones in Backus et al.[3] for relative low values of both k and σ.

Chapter 5

Conclusions

The main purpose of this work was to explain the expectation hypothesis bias through the arbitrage theory, considering two instantaneous spot rate models, Vasicek and CIR.

We showed that the forward rates are unbiased predictors of the future spot rates in a T-forward measure, a probability measure where we define a T-bond as the numeraire. On the other hand, we needed to obtain an unbiased predictor of the instantaneous spot rates in "real world" measure P, since this is the probability measure where prices are observed. Because of this we derived adjustments to allow for an unbiased closed solution for the expected future instantaneous spot rate in P measure.

In particular for the Vasicek and CIR models, their dynamics are defined in the risk neutral probability measure Q. To change their dynamics from the Q measure to P measure we use the same methodology used in Cox et al.[11] using representative agent theory. To approach this change they assumed all agents have homogeneous beliefs represented by a constant relative risk aversion utility function, where the CRRA factor was used as Girsanov kernel. After we were able to obtain closed form solution for the expected instantaneous spot rate in Q and P measures. We used the fact that the expected value of instantaneous spot rates in T-forward measure are the forward rates for our computations. To obtain solution for the forward rates in these models, we have based on the fact that they can be seen as affine term structure. Through this formulation we can obtain closed forms for forward rates.

With all these we were able to compute risk aversion and stochastic adjustments to be added along side forward rates, obtaining an unbiased closed form solution for the expected future spot rates.

With the parameters extracted from Zeytun and Gupta[24] we computed some calculations for both models. In particular, we focus results from subsection 4.3 were we tried to give an answer for the results obtained by Backus et al.[3]. Only for some parameters we were able to explain the high values of CRRA factors found by them. In particular in the Vasicek model only for lower values of k and higher maturities we could find values of ϵ^* greater than 8. For the CIR model only for lower values of k and σ we could find values of ϵ^* greater than 8.

Chapter 6

Appendix

6.1 Vasicek table results

Table 6.1 refers to Vasicek model with parameters $k = 0.5$, $\sigma = 0.01$ and $\theta = 0.1$, varying the constant relative risk aversion factor ϵ.

	Value
r_t	2,5%
θ	0,1
σ	0,01
k	0,5

(a) Parameters Table

	1	2	5	10
f(t,T)	5,45	7,23	9,37	9,93
$E^P[R(T)]$	5,45	7,24	9,38	9,95
SA(t,T)	0	0,01	0,02	0,02
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	0,06	0,11	0,18	0,2
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	5,45	7,23	9,37	9,93
$E^P[R(T)]$	5,46	7,25	9,39	9,96
SA(t,T)	0	0,01	0,02	0,02
RA(t,T)	0	0,01	0,01	0,01
bias weight in E[R(T)]	0,13	0,2	0,28	0,3
SA weight in bias	43,73	55,92	64,84	66,5
RA weight in bias	55,08	44,09	35,4	33,37

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	5,45	7,23	9,37	9,93
$E^P[R(T)]$	5,46	7,25	9,4	9,97
SA(t,T)	0	0,01	0,02	0,02
RA(t,T)	0,01	0,01	0,02	0,02
bias weight in E[r(T)]	0,2	0,29	0,37	0,4
SA weight in bias	28,2	38,63	47,89	49,86
RA weight in bias	71,95	60,9	52,29	50,29

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	5,45	7,23	9,37	9,93
$E^P[R(T)]$	5,47	7,27	9,42	9,99
SA(t,T)	0	0,01	0,02	0,02
RA(t,T)	0,02	0,03	0,04	0,04
bias weight in E[r(T)]	0,35	0,46	0,57	0,6
SA weight in bias	16,4	24,01	31,45	33,18
RA weight in bias	83,16	76	68,48	66,76

(e) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	5,45	7,23	9,37	9,93
$E^P[R(T)]$	5,49	7,3	9,48	10,05
SA(t,T)	0	0,01	0,02	0,02
RA(t,T)	0,04	0,06	0,09	0,1
bias weight in E[r(T)]	0,77	0,97	1,15	1,18
SA weight in bias	7,29	11,23	15,5	16,57
RA weight in bias	92,51	88,78	84,46	83,4

(f) $\epsilon = 1$

Table 6.1: Vasicek ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.01$, k = 0.5

6.2 CIR table results

Table 6.2 refers to CIR model with parameters $k = 0.5$, $\sigma = 0.05$ and $\theta = 0.1$. The difference between the tables is the constant relative risk aversion factor ϵ which varies between 0 and 5.

	Value
r_t	2,5%
θ	0,1
σ	0,05
k	0,5

(a) Parameters Table

	1	2	5	10
f(t,T)	5,45	7,23	9,35	9,9
$E^P[R(T)]$	5,45	7,24	9,38	9,95
SA(t,T)	0	0,01	0,03	0,05
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	0,05	0,14	0,34	0,47
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	5,45	7,23	9,35	9,9
$E^P[R(T)]$	5,46	7,25	9,4	9,97
SA(t,T)	0	0,01	0,03	0,05
RA(t,T)	0	0,01	0,02	0,02
bias weight in E[R(T)]	0,13	0,26	0,55	0,71
SA weight in bias	40,92	52,44	62,74	65,92
RA weight in bias	59,08	47,56	37,26	34,08

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	5,45	7,23	9,35	9,9
$E^P[R(T)]$	5,46	7,26	9,42	10
SA(t,T)	0	0,01	0,03	0,05
RA(t,T)	0,01	0,02	0,04	0,05
bias weight in E[r(T)]	0,2	0,38	0,75	0,96
SA weight in bias	25,9	35,41	45,67	49,09
RA weight in bias	74,1	64,59	54,33	50,91

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	5,45	7,23	9,35	9,9
$E^P[R(T)]$	5,47	7,28	9,46	10,05
SA(t,T)	0	0,01	0,03	0,05
RA(t,T)	0,02	0,04	0,08	0,1
bias weight in E[R(T)]	0,36	0,63	1,15	1,44
SA weight in bias	14,78	21,51	29,52	32,42
RA weight in bias	85,22	78,49	70,48	67,58

(e) $\epsilon = 2$

	1	2	5	10
f(t,T)	5,45	7,23	9,35	9,9
$E^P[R(T)]$	5,49	7,33	9,58	10,2
SA(t,T)	0	0,01	0,03	0,05
RA(t,T)	0,04	0,09	0,19	0,25
bias weight in E[R(T)]	0,81	1,36	2,37	2,89
SA weight in bias	6,49	9,84	14,22	15,92
RA weight in bias	93,51	90,16	85,78	84,08

(f) $\epsilon = 5$

Table 6.2: CIR ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k = 0.5

Table 6.3 refers to CIR model with parameters $k = 0.25$, $\sigma = 0.05$ and $\theta = 0.2$. The difference between the tables is the constant relative risk aversion factor ϵ which varies between 0 and 5.

	Value
r_t	2,5%
θ	0,2
σ	0,05
k	0,25

(a) Parameters Table

	1	2	5	10
f(t,T)	6,37	9,37	14,89	18,32
$E^P[R(T)]$	6,37	9,39	14,99	18,56
SA(t,T)	0	0,02	0,1	0,24
RA(t,T)	0	0	0	0
bias weight in E[r(T)]	0,06	0,18	0,64	1,3
SA weight in bias	100	100	100	100
RA weight in bias	0	0	0	0

(b) $\epsilon = 0$

	1	2	5	10
f(t,T)	6,37	9,37	14,89	18,32
$E^P[R(T)]$	6,38	9,4	15,03	18,64
SA(t,T)	0	0,02	0,1	0,24
RA(t,T)	0,01	0,01	0,04	0,07
bias weight in E[R(T)]	0,14	0,31	0,9	1,69
SA weight in bias	43,08	56,58	70,37	76,5
RA weight in bias	56,92	43,42	29,63	23,5

(c) $\epsilon = 0.5$

	1	2	5	10
f(t,T)	6,37	9,37	14,89	18,32
$E^P[R(T)]$	6,38	9,41	15,07	18,71
SA(t,T)	0	0,02	0,1	0,24
RA(t,T)	0,01	0,03	0,08	0,15
bias weight in E[r(T)]	0,22	0,45	1,16	2,08
SA weight in bias	27,47	39,35	54,25	61,86
RA weight in bias	72,53	60,65	45,75	38,14

(d) $\epsilon = 1$

	1	2	5	10
f(t,T)	6,37	9,37	14,89	18,32
$E^P[R(T)]$	6,39	9,44	15,15	18,86
SA(t,T)	0	0,02	0,1	0,24
RA(t,T)	0,02	0,05	0,16	0,3
bias weight in E[R(T)]	0,38	0,72	1,69	2,87
SA weight in bias	15,93	24,45	37,12	44,63
RA weight in bias	84,07	75,55	62,88	55,37

(e) $\epsilon = 2$

	1	2	5	10
f(t,T)	6,37	9,37	14,89	18,32
$E^P[R(T)]$	6,42	9,52	15,39	19,33
SA(t,T)	0	0,02	0,1	0,24
RA(t,T)	0,05	0,13	0,41	0,76
bias weight in E[R(T)]	0,86	1,53	3,27	5,2
SA weight in bias	7,04	11,42	18,91	24,01
RA weight in bias	92,96	88,58	81,09	75,99

(f) $\epsilon= 5$

Table 6.3: CIR ϵ calculations for parameters: $r_t = 2.5\%$, $\theta = 0.2$, $\sigma = 0.05$, k = 0.25

6.3 Vasicek implicit risk aversion results

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,4877	1,4877	2,4877	3,4877	4,4877	5,4877
T=2	0,9516	1,9516	2,9516	3,9516	4,9516	5,9516
T=5	2,212	3,212	4,212	5,212	6,212	7,212
T=10	3,9347	4,9347	5,9347	6,9347	7,9347	8,9347
T=20	6,3212	7,3212	8,3212	9,3212	10,3212	11,3212

Table 6.4: Vasicek: Values of ϵ^* considering k = 0.05

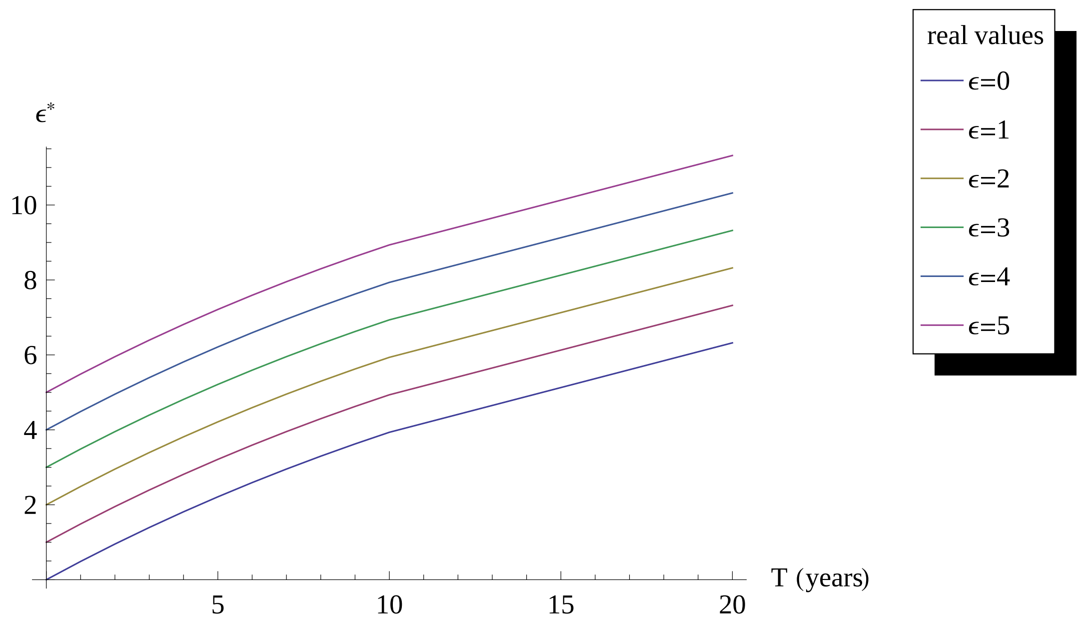

Figure 6.1: Graphic of ϵ^* considering that bias is only risk aversion effect: k=0.05

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,3935	1,3935	2,3935	3,3935	4,3935	5,3935
T=2	0,6321	1,6321	2,6321	3,6321	4,6321	5,6321
T=5	0,9179	1,9179	2,9179	3,9179	4,9179	5,9179
T=10	0,9933	1,9933	2,9933	3,9933	4,9933	5,9933
T=20	0,9999	1,9999	2,9999	3,9999	4,9999	5,9999

Table 6.5: Vasicek: Values of ϵ^* considering k = 0.5

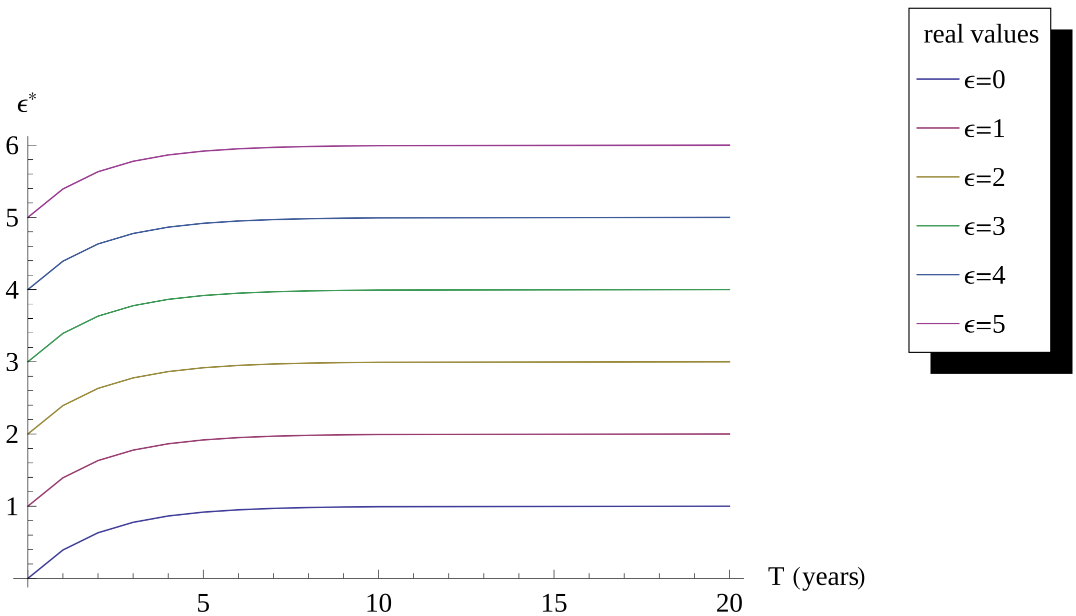

Figure 6.2: Graphic of ϵ^* considering that bias is only risk aversion effect: k=0.5

6.4 CIR implicit risk aversion results

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,4635	1,4624	2,4613	3,4602	4,4591	5,458
T=2	0,8688	1,865	2,8611	3,8572	4,8534	5,8495
T=5	1,8561	2,8374	3,8188	4,8004	5,7821	6,7641
T=10	3,0108	3,9565	4,903	5,8504	6,7986	7,7476
T=20	4,2264	5,0953	5,9675	6,8429	7,7214	8,603

Table 6.6: CIR: Values of ϵ^*: $r_t = 2.5\%$, $\theta = 0.2$, $\sigma = 0.05$, k $= 0.05$

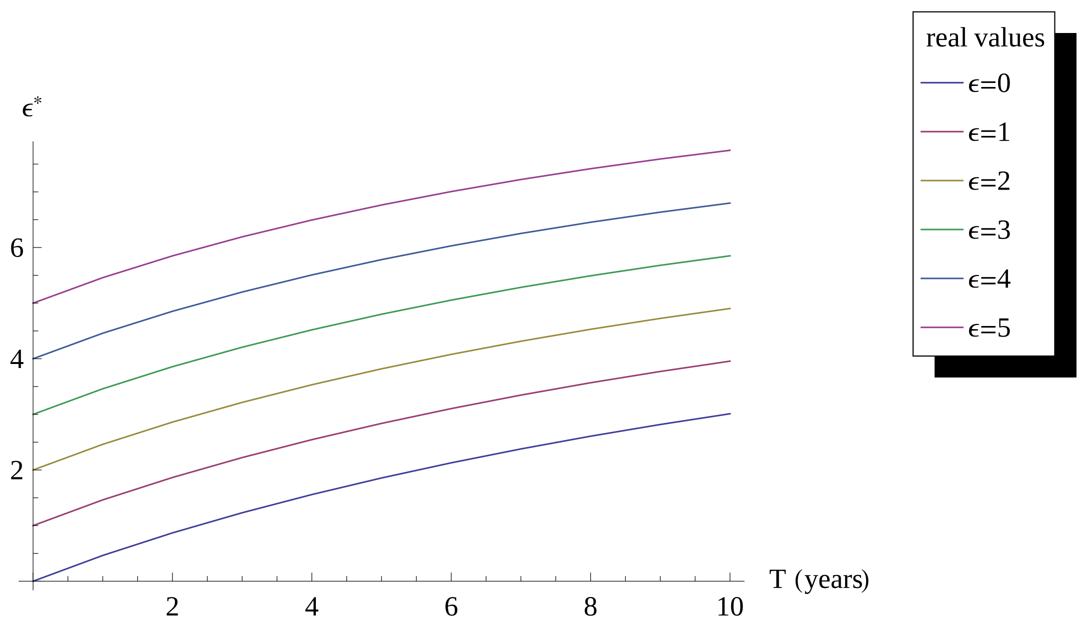

Figure 6.3: Graphic of ϵ^* considering that bias is only risk aversion effect: $r_t = 2.5\%$, $\theta = 0.2$, $\sigma = 0.05$, k $= 0.05$

	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = 2$	$\epsilon = 3$	$\epsilon = 4$	$\epsilon = 5$
T=1	0,4062	1,4053	2,4044	3,4035	4,4026	5,4017
T=2	0,6949	1,6921	2,6894	3,6866	4,6838	5,6811
T=5	1,234	2,2237	3,2135	4,2033	5,1932	6,1831
T=10	1,6418	2,6198	3,5979	4,5763	5,5549	6,5337
T=20	1,8562	2,8224	3,7889	4,7558	5,7231	6,6908

Table 6.7: CIR:l Values of ϵ^*: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k $=$ 0.25

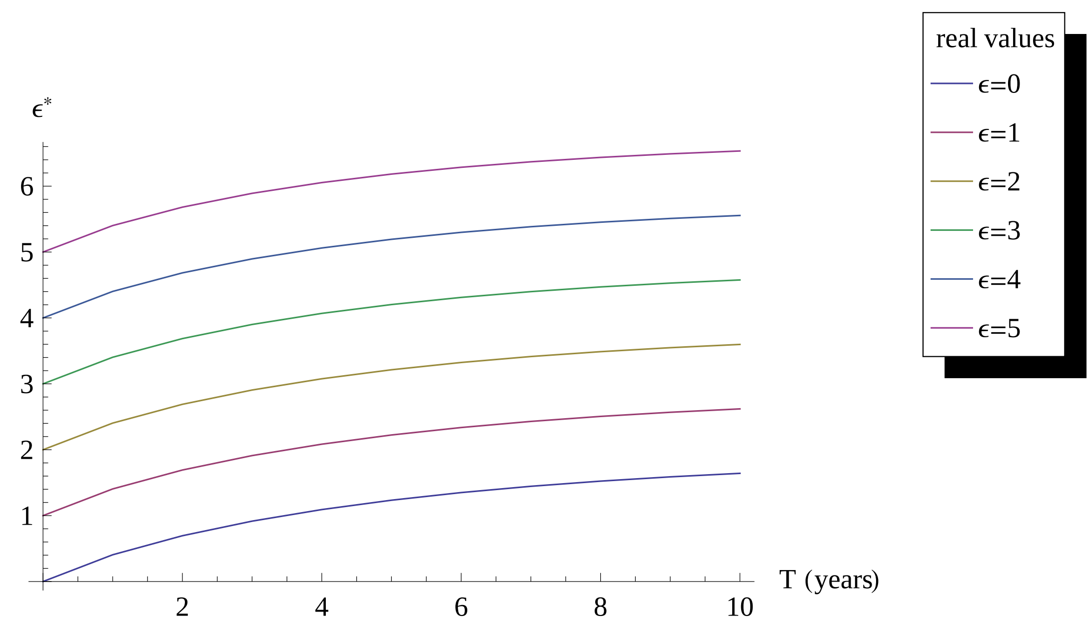

Figure 6.4: Graphic of ϵ^* considering that bias is only risk aversion effect: $r_t = 2.5\%$, $\theta = 0.1$, $\sigma = 0.05$, k $= 0.25$

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